Comparison principles for variational problems

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Goal

Given two parameters or data of a problem f_1, f_2, let $E(\cdot, f_1), E(\cdot, f_2)$ be two functionals on X. We want to compare the solutions

$$u(f_1) \in \operatorname*{arg\,min}_{u \in X} E(u, f_1) \quad \text{and } u(f_2) \in \operatorname*{arg\,min}_{u \in X} E(u, f_2).$$

Under which conditions do we have $u(f_1) \leq u(f_2)$?

Examples of applications

$\mathcal{T}(\mu, \nu)$ is a transport cost: standard, entropic or unbalanced OT.

- Comparison principles for JKO type problems:

Theorem

$S(\mu) = \arg\min_{\nu \in \mathcal{M}_+(\Omega^*)} \mathcal{T}(\mu, \nu) + H(\nu)$ is monotone:

$$\mu_1 \leq \mu_2 \implies S(\mu_1) \leq S(\mu_2)$$

Here

$$E(\nu, \mu_1) = \mathcal{T}(\mu_1, \nu) + H(\nu), \quad E(\nu, \mu_2) = \mathcal{T}(\mu_2, \nu) + H(\nu).$$

Examples of applications

Let $\Phi(\mu,\nu) = \arg\max_\phi \int \phi d\nu - \int (\phi)^c d\mu_i$ be the set of Kantorovich potentials.

- Comparison principles for Kantorovich potentials:

Theorem

Let $\phi_i \in \Phi(\mu_i,\nu)$ and U an open subset of Ω

$$\begin{cases} \mu_1 \leq \mu_2 & \text{on } U \\ \phi_1 \leq \phi_2 & \text{on } \Omega \setminus U \end{cases} \implies \begin{cases} \phi_1 \wedge \phi_2 \in \Phi(\mu_1,\nu) \\ \phi_1 \vee \phi_2 \in \Phi(\mu_2,\nu). \end{cases}$$

Here

$$E(\phi,\mu_1) = \int \phi d\nu - \int (\phi)^c d\mu_1, \quad E(\phi,\mu_2) = \int \phi d\nu - \int (\phi^c) d\mu_2.$$

with $\phi^c(y) = \sup_x \phi(x) - c(x,y)$.

Why comparison principles ?

- **Control of the minimizers**: Given f if $u(f_0)$ is easy to compute for $f \leq f_0$ then $u(f) \leq u(f_0)$.
- **Contractivity in L^1/L^∞ [Crandall and Tartar, 1980]:** if $f \mapsto u(f)$ preserves the mass and the order then

$$\|u(f_1) - u(f_2)\|_{L^1} \leq \|f_1 - f_2\|_{L^1}.$$

Method to derive comparison principles

First : **Compare functionals:** $E_1 \ll E_2$.

Second : **Compare minimizers:** $E_1 \ll E_2 \implies \arg\min E_1 \ll \arg\min E_2$.

(Third) : **Compare singletons:** $\{u_1\} \ll \{u_2\} \iff u_1 \leq u_2$.

How to define $\ll$ on the functions as well as sets ?

Submodularity and comparison

History of submodularity

A function $E : \mathbb{R}^d \to \mathbb{R}$ is *submodular* if:

$$E(x \wedge y) + E(x \vee y) \leq E(x) + E(y)$$

where $x \wedge y = (\min(x_i, y_i))_i$ and $x \vee y = (\max(x_i, y_i))_i$.

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Capacity theory [Choquet, 1953] to study capacitable sets,

Economics [Lorentz, 1953, Topkis, 1978]: monotone comparative statics in $\mathbb{R}^d$ and in lattices,

Optimization Combinatorial optimization [Murota, 2003, Lee, 2004], review of submodularity in $\mathbb{R}^d$ [Bach, 2018],

Calculus of variations [Chambolle and Darbon, 2009] to derive comparison principles for optimizers of the perimeter.

Examples of lattices

Spaces where $\wedge, \vee$ are well defined are called lattices. Here are some examples:

- $\mathbb{R}^n$ where $x \leq y \iff x_i \leq y_i$ for all i.
- **Sobolev space:** If $U \subset \mathbb{R}^n$ is open then $H^1(U)$, with $u \leq v$ if $u(x) \leq v(x)$ for almost every $x \in U$, is a lattice.

$$u \vee v(x) = u(x) \vee v(x), \nabla(u \vee v)(x) = \nabla u(x) \chi_{u(x) \geq v(x)} + \nabla v(x) \chi_{u(x) < v(x)}$$

- **Continuous functions:** $C(\Omega)$ is a lattice with $\phi \leq \psi$ if $\phi(x) \leq \psi(x)$ for all $x \in \Omega$.
- **Radon measures:** $\mathcal{M}(\Omega)$ is a lattice with

$$\mu \leq \nu \iff \forall \phi \in C(\Omega), \phi \geq 0, \int \phi d\mu \leq \int \phi d\nu$$

Examples of submodular functionals

- **Dirichlet's energy** on $H^1(U)$

$$u \mapsto \int_U \|\nabla u(x)\|^2 dx.$$

- **Kantorovich** functional in optimal transport

$$(\phi \in C(\Omega)) \mapsto \int \phi^c d\nu$$

where $\nu \in M_+(\Omega)$ and $\phi^c(y) = \sup_x \phi(x) - c(x, y)$.

First: Define $\ll$ on functions

X is a Banach space equipped with a preorder $\leq$ such that $(X, \leq)$ is a lattice.
Recall that E is submodular if for any $\phi, \psi \in X$:

$$E(\phi \wedge \psi) + E(\phi \vee \psi) \leq E(\phi) + E(\psi)$$

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We say that E_1 is P-dominated by E_2 ($E_1 \ll_{\mathrm{P}} E_2$) if :

$$E_1(\phi_1 \wedge \phi_2) + E_2(\phi_1 \vee \phi_2) \leq E_1(\phi_1) + E_2(\phi_2).$$

Observe that if $E_1 \ll_{\mathrm{P}} E_2, E_3 \ll_{\mathrm{P}} E_4$, then $E_1 + E_3 \ll_{\mathrm{P}} E_2 + E_4$

First: Define $\ll$ on sets

Let A_1, A_2 be two subsets of X.
We say that A_1 is P-dominated by A_2 ($A_1 \ll_{\mathrm{P}} A_2$) if

$$\begin{cases} \phi_1 \in A_1 \\ \phi_2 \in A_2 \end{cases} \implies \begin{cases} \phi_1 \wedge \phi_2 \in A_1 \\ \phi_1 \vee \phi_2 \in A_2. \end{cases}$$

This is equivalent to the P-dominance of their convex indicator functions: $\iota_{A_1} \ll_{\mathrm{P}} \iota_{A_2}$.

Second: Compare minimizers

Theorem [Topkis, 1978]

Let $E_1, E_2 \colon X \to \mathbb{R}$. If $E_1 \ll_{\mathrm{P}} E_2$ then

$$\arg\min E_1 \ll_{\mathrm{P}} \arg\min E_2.$$

Take $\phi_1 \in \arg\min E_1$ and $\phi_2 \in \arg\min E_2$ then by P-dominance we have

$$E_1(\phi_1 \wedge \phi_2) + E_2(\phi_1 \vee \phi_2) \leq E_1(\phi_1) + E_2(\phi_2).$$

Which implies that $\phi_1 \wedge \phi_2 \in \arg\min E_1$ and $\phi_1 \vee \phi_2 \in \arg\min E_2$.

Third: Compare singletons

Lemma

Let $\phi_1, \phi_2 \in X$ then $\{\phi_1\} \ll_{\mathrm{P}} \{\phi_2\}$ if and only if $\phi_1 \leq \phi_2$.

This approach recovers comparison principles on the (fractional) laplacian, minimizers of the perimeter [Chambolle and Darbon, 2009] and Monge Ampère equation in OT.

Examples of ordered functionals

Let Ω be a compact metric space. Let $\mu_1, \mu_2 \in \mathcal{M}_+(\Omega)$ be such that $\mu_1 \leq \mu_2$.

- $-\langle \cdot, \mu_1 \rangle \ll_{\mathrm{P}} -\langle \cdot, \mu_2 \rangle$ on $C(\Omega)$.
- The Kantorovich functionals satisfy $K_{\mu_2} \ll_{\mathrm{P}} K_{\mu_1}$ on $C(\Omega)$.

$$\int (\phi_1 \vee \phi_2)^c d\mu_1 + \int (\phi_1 \wedge \phi_2)^c d\mu_2 \leq \int (\phi_1)^c d\mu_1 + \int (\phi_2)^c d\mu_2.$$

Proof of $K_{\mu_2} \ll_{\mathrm{P}} K_{\mu_1}$

Let $x \in \Omega$ then

$$(\phi_1 \vee \phi_2)^c(x) + (\phi_1 \wedge \phi_2)^c(x) = \sup_y \phi_1(y) \vee \phi_2(y) - c(x,y) + \sup_z \phi_1(z) \wedge \phi_2(z) - c(x,z)$$

$$= \sup_y \max(\phi_1(y) - c(x,y), \phi_2(y) - c(x,y)) + \sup_z \min(\phi_1(z) - c(x,z), \phi_2(z) - c(x,z))$$

$$\leq (\phi_1)^c \vee (\phi_2)^c + (\phi_1)^c \wedge (\phi_2)^c = (\phi_1)^c + (\phi_2)^c.$$

The result follows by integrating against μ_1 and μ_2 and using $\mu_1 \leq \mu_2$.

Duality ?

The dual functionals in OT, entropic OT and unbalanced OT are all submodular and ordered if μ varies.
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For comparison on the space of measures we have the following issue:

$$\mathcal{T}(\mu_i,\nu) = \sup_{\phi\in C(\Omega)} \int \phi d\nu - K_{\mu_i}(\phi) = K^*_{\mu_i}(\nu).$$

Question If $K_{\mu_2} \ll_{\mathrm{P}} K_{\mu_1}$ then $K^*_{\mu_1} \leq_? K^*_{\mu_2}$?

Substitutability and comparison

Duality and Substitutability

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Definition of $\ll_{\mathrm{Q}}$

We say that F_1 is Q-dominated by F_2 ($F_1 \ll_{\mathrm{Q}} F_2$) if for any $\mu_1, \mu_2 \in X^*$:
$\forall t_{12},\ 0 \leq t_{12} \leq (\mu_1 - \mu_2)^+, \qquad \exists t_{21},\ 0 \leq t_{21} \leq (\mu_1 - \mu_2)^-$

$$F_1(\mu_1 - (t_{12} - t_{21})) + F_2(\mu_2 + (t_{12} - t_{21})) \leq F_1(\mu_1) + F_2(\mu_2).$$

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$$F_1(\mu_1 - (t_{12} - t_{21})) + F_2(\mu_2 + (t_{12} - t_{21})) \le F_1(\mu_1) + F_2(\mu_2).$$

If $F \ll_{\mathrm{Q}} F$ then F is said to be substitutable. Substitutability is stronger than convexity. Variations of this definition: $M^\natural$-convexity in discrete convex analysis [Murota, 2003], S-convexity on $\mathbb{R}^n$ [Chen and Li, 2020].

Second: compare minimizers

Theorem

If $F_1 \ll_{\mathrm{Q}} F_2$ then

$$\arg\min F_1 \ll_{\mathrm{Q}} \arg\min F_2.$$

Take $\mu_1 \in \arg\min F_1$ and $\mu_2 \in \arg\min F_2$ then by Q-dominance we have

$$F_1(\mu_1 - (t_{12} - t_{21})) + F_2(\mu_2 + (t_{12} - t_{21})) \leq F_1(\mu_1) + F_2(\mu_2).$$

Which implies that $\mu_1 - (t_{12} - t_{21}) \in \arg\min F_1$ and $\mu_2 + (t_{12} - t_{21}) \in \arg\min F_2$.

Third: compare singletons

Lemma

Let $\mu_1, \mu_2 \in X^*$ then $\{\mu_1\} \ll_{\mathrm{Q}} \{\mu_2\}$ if and only if $\mu_1 \leq \mu_2$.

$\mu_1 - (t_{12} - t_{21}) \in \{\mu_1\}$ forces $t_{12} = 0$ but it is chosen freely in $0 \leq t_{12} \leq (\mu_1 - \mu_2)^+$.
Thus $(\mu_1 - \mu_2)^+ = 0$ i.e. $\mu_1 \leq \mu_2$.

Examples of Q-ordered functionals

Ω is a compact metric space. Let $\mu_1, \mu_2 \in \mathcal{M}_+(\Omega)$ be such that $\mu_1 \leq \mu_2$.
Since $K_{\mu_2} \ll_{\mathrm{P}} K_{\mu_1}$ we have by duality

$$\mathcal{T}(\mu_1, \cdot) \ll_{\mathrm{Q}} \mathcal{T}(\mu_2, \cdot)$$

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For JKO scheme we want to compare

$$\arg\min_{\nu} \mathcal{T}(\mu_i, \nu) + H(\nu), \quad i = 1, 2.$$

Q-dominance and sum

However $H \ll_{\mathrm{Q}} H$ (H substitutable) is not enough to imply $F_1 + H \ll_{\mathrm{Q}} F_2 + H$.

Proposition

If $F_1 \ll_{\mathrm{Q}} F_2$ and H is separable convex then $F_1 + H \ll_{\mathrm{Q}} F_2 + H$.

Internal energies $H_{f,m} : \nu \mapsto \int f(\frac{d\nu}{dm})dm$ where $f : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is a proper convex function are totally substitutable.

Application to Optimal transport

Reminder OT

For $\mu, \nu \in \mathcal{M}_+(\Omega)$ such that $\mu(\Omega) = \nu(\Omega)$, the entropic optimal transport problem is

$$OT_{c,\epsilon}(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int c d\pi + \epsilon \, \mathrm{KL}(\pi \mid \alpha \otimes \beta) = \sup_{\phi \in C(\Omega)} \int \phi d\nu - K_{\epsilon,\mu}(\phi)$$

where $\mathrm{KL}(\pi \mid \alpha \otimes \beta) = \int \log(\frac{d\pi}{d\alpha \otimes d\beta}) d\pi$ and

$$K_{\epsilon,\mu}(\phi) = \int \epsilon \log\left(\int e^{[\phi(y) - c(x,y)]/\epsilon} d\beta(y)\right) d\mu(x), \quad K_{0,\mu}(\phi) = \int \phi^c d\mu.$$

Comparison principle for JKO

Theorem

Let $c \in C(\Omega \times \Omega)$ and $h\colon [0, +\infty) \to \mathbb{R}$ denote a strictly convex l.s.c. and superlinear function. Define $H_{f,m}\colon (\nu \in \mathcal{M}_+(\Omega)) \mapsto \int f(\frac{d\nu}{dm})\, dm$ where $m \in \mathcal{M}_+(\Omega)$ is a fixed reference measure. If $\mu_1 \leq \mu_2$ then

$$\arg\min_{\nu} OT_{c,\epsilon}(\mu_1, \nu) + H_{f,m}(\nu) \leq \arg\min_{\nu} OT_{c,\epsilon}(\mu_2, \nu) + H_{f,m}(\nu)$$

[Jacobs et al., 2020] derived this result for the standard OT ($\epsilon = 0$) with a regular cost c such that the optimal transport map exists.
In a translation invariant setting the L^1 contraction grants bound in BV.

Proof of comparison principle for JKO

1. $K_{\varepsilon,\mu_2} \ll_{\mathrm{P}} K_{\varepsilon,\mu_1}$ on $C(\Omega)$,
2. $K^*_{\varepsilon,\mu_1} \ll_{\mathrm{P}} K^*_{\varepsilon,\mu_2}$, that is $OT_{c,\epsilon}(\mu_1,\cdot) \ll_{\mathrm{Q}} OT_{c,\epsilon}(\mu_2,\cdot)$.
3. $H_{f,m}$ is convex separable thus $OT_{c,\epsilon}(\mu_1,\cdot) + H_{f,m} \ll_{\mathrm{Q}} OT_{c,\epsilon}(\mu_2,\cdot) + H_{f,m}$.
4. $\arg\min OT_{c,\epsilon}(\mu_1,\cdot) + H_{f,m} \ll_{\mathrm{Q}} \arg\min OT_{c,\epsilon}(\mu_2,\cdot) + H_{f,m}$.
5. Conclude with the uniqueness of the minimizers due to the strict convexity of $H_{f,m}$.

Conclusion

- Time evolution. $u \in \arg\min E$

$$\nabla E(u) = 0 \to \partial_t u + \nabla E(u) = 0$$

- Leave the variational framework: off-diagonal antitone [Ortega and Rheinboldt, 1970].
- Beyond pointwise order: subharmonic order.

Thank you for your attention !

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