Stable matching with non transferrable utility

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Stable Matching Problem (linear transfers)

Two sided population with types $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. The number of agent is infinite and n_x, m_y represent the proportion of a type in their respective side. We look for a matching outcome $(\mu, u, v) \in \mathbb{R}_+^{X \times Y} \times \mathbb{R}^X \times \mathbb{R}^X$ that satisfies

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Feasibility : $\sum_y \mu_{xy} = n_x$ and $\sum_x \mu_{xy} = m_y$

Stability : $u_x + v_y \geq \phi_{xy}$

No slackness : If $\mu_{xy} > 0$ then $u_x + v_y = \phi_{xy}$

Following [GS15] we will rephrase this problem in terms of generalized equilibrium matchings.

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Introduce U and V the endogeneous utilities.Then it is the equivalent to find (μ, U, V) such that

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Introducing $G^*_X(\mu) = 1_{\sum_y \mu_{xy} = n_x} + 1_{\mu \geq 0}$ (and symmetrically for Y) allows you to rewrite the feasability condition.

Stable Matching Problem (linear transfers)

We search for (μ, U, V) satisfying:

$$\text{Feasibility} : \mu \in \partial G_X(U) \cap \partial G_Y(V)$$
$$\text{Stability} : U_{xy} + V_{xy} = \phi_{xy}$$

Where

$$G_X(U) = \sum_x n_x \sup_y U_{xy}$$

is the aggregated welfare function. Thus feasibility reinterprets as matching demand and offer under endogeneous utilities (U, V).
Note that you recover μ the solution of the OT problem.

$$\sup_{\mu \in \Pi(n,m)} \sum_{xy} \phi_{xy} \mu_{xy}$$

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If you set $G_X^*(\mu) = 1_{\sum_y \mu_{xy} = n_x} + 1_{\mu \geq 0} + \sum_{xy} \mu_{xy} \log(\mu_{xy})$ you get the entropy regularised optimal transport:

$$\sup_{\mu \in \Pi(n,m)} \sum_{xy} \phi_{xy} \mu_{xy} - \sum_{xy} \mu_{xy} \log(\mu_{xy})$$

Stable Matching Problem : NTU

Given two convex functions G_X, G_Y and two preferences vectors α, γ. We search for an equilibrium matching, that is a triplet (μ, U, V) satisfying:

Feasibility : $\mu \in \partial G_X(U) \cap \partial G_Y(V)$

Stability : $\min(\alpha_{xy} - U_{xy}, \gamma_{xy} - V_{xy}) = 0$

The two quantities $\tau^X_{xy} = \alpha_{xy} - U_{xy}, \tau^Y_{xy} = \gamma_{xy} - V_{xy}$ represent the waiting times on each side of the market.

When does such an equilibrium matching exist?

In the discrete setting (i.e. $\mu_{xy} \in \{0, 1\}$) [GS62] proved existence and gave an algorithm to find the stable matching.

Structure of the feasible matchings

When $G_X(U) = \sum_x n_x \sup_y U_{xy}$ (i.e. OT) then G_X is submodular that is

$$G_X(U \wedge U') + G_X(U \vee U') \leq G_X(U) + G_X(U')$$

In economics this submodular property of the aggregated welfare function is gross substituability [Pae17, GSV22].

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What can we say about the subdifferential

$$\partial G_X(U) = \arg\max_{\mu} \sum_{xy} \mu_{xy} U_{xy} - G_X^*(\mu)$$

of submodular functions?
Monotone comparative statics exist in the discrete case [Vei89, HM05] or for supermodular functions [Top98, MS94, Ech02, QS09].

Orders on functions/sets - P order

For any functions f, g we introduce the P-order and the Q-order.

P-order

We say that f is smaller than g in the P-order, $f \leq_{P-order} g$, if for any U, U' we have

$$f(U \wedge U') + g(U \vee U') \leq f(U) + g(U')$$

f is submodular if $f \leq_{P-order} f$

When f, g are indicator functions of sets the P-order is Veinott's strong set order.

Orders on functions/sets - Q order

Q-order

We say that f is smaller than g in the Q-order, $f \leq_{Q-order} g$, if if for any $\mu, \mu' \in \mathbb{R}^{X \times Y}$ and any $\delta_1 \in [0, (\mu - \mu')^+]$ there is $\delta_2 \in [0, (\mu - \mu')^-]$ such that

$$f(\mu - (\delta_1 - \delta_2)) + g(\mu' + (\delta_1 - \delta_2)) \leq f(\mu) + g(\mu')$$

we say that f is exchangeable if $f \leq_{Q-order} f$.

Exchangeability extends $M^\natural$-convexity [Mur98] to non polyhedral functions. The Q-order extends the notion of S-convexity [CL23].

Examples of P and Q ordered sets

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Monotone comparative statics

Theorem

Let f a convex (closed, proper, lsc) function. Then the following assertions are equivalent.

- f is submodular
- f^* is exchangeable
- For $U \leq U'$, $\pi_{U=U'} \partial f(U') \leq_{Q-order} \pi_{U=U'} \partial f(U)$

If any one them is satsified then

$$\bar{\mu} - \underset{\mu \leq \bar{\mu}}{\arg\max} \sum_{xy} \mu_{xy} U_{xy} - f^*(\mu)$$

is increasing in the Q-order in $\bar{q}$.

Remarks

- If f is differentiable the Q-order property simply reads as

$$(\nabla f(U'))_{xy} \leq (\nabla f(U))_{xy}$$

  if $U' \geq U$ and $U_{xy} = U'_{xy}$.
- If f is twice differentiable then we recover the C^2 characterisation of submodularity of f

$$\frac{\partial^2 f}{\partial U_{xy} \partial U_{x'y'}} \leq 0$$

Continuous time Gale & Shapley algorithm

At step 0, we set $\mu^{A,0} = +\infty$ and $\mu^{T,-1} = 0$.

Proposal phase: The x's make proposals to the y's subject to availability constraint:

$$\mu^{P,k} \in \underset{\mu^{T,k-1} \leq \mu \leq \mu^{A,k}}{\arg\max} \; \mu^\top \alpha - G^*(\mu),$$

Disposal phase: The the y's pick their best offers among the proposals:

$$\mu^{T,k} \in \underset{\mu \leq \mu^{P,k}}{\arg\max} \, \mu^\top \gamma - H^*(\mu)$$

Update phase: The number of available offers is decreased by the number of rejected ones:

$$\mu^{A,k+1} = \mu^{A,k} - \left(\mu^{P,k} - \mu^{T,k}\right).$$

Existence of equilibrium matching

Definition

$G : \mathbb{R}^{X \times Y} \to \mathbb{R}$ is a welfare function if G is convex (closed proper lsc), $\operatorname{dom} G^*$ is compact and $0 = \min(\operatorname{dom} G^*)$.

Theorem

Assume G_X, G_Y are submodular welfare functions then for any initial preferences α, γ there exists an equilibrium matching (μ, U, V):

- $\mu \in \partial G_X(U) \cap \partial G_Y(V)$
- $\min(\alpha_{xy} - U_{xy}, \gamma_{xy} - V_{xy}) = 0$

Application to the random utility model [McF77]

For any collection $(\mathbf{P}_x)$ of probability measures on $\mathbb{R}^Y$ the random utility welfare function

$$G_X(U) = \sum_{x \in X} n_x \mathbb{E}_{\mathbf{P}_x} \left[\max_{y \in Y} \{ U_{xy} + \varepsilon_y, \varepsilon_0 \} \right]$$

is a submodular welfare function. Thus an equilibrium matching exists.

If $\mathbf{P}_x$ is a the law of $|Y|$ i.i.d. Gumbel distributions then

$$G_X(U) = \sum_{x \in X} n_x \log(1 + \sum_{y \in Y} \exp U_{xy})$$

Ending remarks

Conclusion

- Introduced a dual version of submodularity which is exchangeability.
- Derived monotone comparative statics for submodular functions.
- Extended Gale & Shapley algorithm to set valued choice function for divisible goods with an application to random utility models.

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Perspectives

- Can we replace subdifferentials by correspondences ?
- From NTU to more general conditions

$$\min(\alpha - U, \gamma - V) = 0 \quad \rightarrow \quad \omega U + V = \Phi?, \quad F(U, V)?$$

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Thank you for your attention!

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